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FINANCIAL ECONOMICS
2 ° YEAR BIEF

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FINANCIAL ECONOMICS

Time Value of Money

IRR➔ Is the only yield that solves the equation of the PV because yields tend to change

$$PV\ annuity = C \times \frac{1-(1+i)^{-t}}{i}$$

$$FV\ annuity = C \times \frac{(1+i)^{t}-1}{i}$$

Statistics Review

- ***Random Variable*** is a value representing an outcome of an uncertain event, whose outcome may be either:
    - Discreet (Scenario analysis)
    - Continuous

- ***Distribution*** is the likelihood of each possible event:
    - For a *discreet outcome*, to each possible scenario we assign a certain probability
    - For a *continuous scenario*, we use the Normal Distribution

- The ***expected value*** is the average outcome of an event, if it was repeated infinitely many times:
    - It is a probability weighted average of the possible outcomes

➔ Suppose the return *Ri* on an asset *i* is equal to *Ri (s),* with probability p(s) for s=1,2,…,s
Then, the **expected return** is:

$$E(R_i) = \sum_{s=1}^{S} R_i(s)p(s)$$

- The ***variance*** is the average squared deviation from the expected value

$$\sigma^2 = E[(Ri(S) - E(Ri)^2] = \sum_{S=1}^{S}[Ri(S) - E(Ri)]^2 \times P(S)$$

- The ***standard deviation*** is the square root of the variance:

$$\sigma = \sqrt{\sigma^2}$$

- The **covariance** between two random variables is the average of the products of their deviations from the mean:

$$\begin{aligned}\text{cov}(R_i, R_j) &= E([R_i - E(R_i)][R_j - E(R_j)]) \\ &= \sum_{s=1}^{S}[R_i(s) - E(R_i)][R_j(s) - E(R_j)]p(s)\end{aligned}$$

➔ The covariance can be:
- **Positive** if one variable tends to be high when the other one is high
- **Negative** if one variable tends to be high when the other is low

➢ The **linear correlation coefficient** is the Covariance between two random variables, divided by the product of their standard deviations:

$$\rho_{ij} = \frac{\text{cov}(R_i, R_j)}{\sigma_i \sigma_j} \qquad -1 \le \rho_{ij} \le 1$$

➢ Mean, Variance, and Covariance can be estimated using historical data and the "sample approach":

$$\hat{E}(R_i) = \frac{1}{T}\sum_{t=1}^{T} R_i(t)$$

$$\hat{\sigma}_i^2 = \frac{1}{T-1}\sum_{t=1}^{T}[R_i(t) - \hat{E}(R_i)]^2$$

$$\hat{\text{cov}}(R_i, R_j) = \frac{1}{T-1}\sum_{t=1}^{T}[R_i(t) - \hat{E}(R_i)][R_j(t) - \hat{E}(R_j)]$$

Portfolio

➢ A **portfolio** is a combination of N assets, with returns $R_1, \dots, R_N$
- In a portfolio, each asset has a weight $\omega_1, \dots, \omega_N$

$$\omega_i = \frac{\$\, value\ of\ stock\ i's\ position}{\$value\ of\ the\ whole\ portfolio}$$

- The sum of all the weights of a portfolio must be 1
- A negative weight indicates a short position

➢ The **return** of a portfolio is the sum of each return times its weight:

$$R_p = \sum_{i=1}^{N} \omega_i \times R_i$$

➢ The **expected return** of a portfolio is the sum of each weight times the expected return of each asset:

$$E(R_p) = \sum_{i=1}^{N} \omega_i \times E(R_i)$$

➢ Given a portfolio with only 2 securities, the portfolio variance is:

$$\sigma_p^2 = \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\rho_{1,2}\sigma_1\sigma_2$$

➔ In general, the **portfolio variance** is:

$$\sigma_p^2 = \sum_{i=1}^{N} \omega_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j>1}^{N} \omega_i\omega_j\rho_{i,j}\sigma_i\sigma_j$$

Return Measures

- *Annual Percentage Rate (APR)* is the interest rate annualized using the simple interest rate and is not compounded

- *Effective Annual Rate (EAR)* is the interest rate annualized using compound rather than simple interest

➔ We can build an equality to compare APR and EAR:

$$EAR = \left(1 + \frac{APR}{n}\right)^n - 1$$

In this case, n is the number of times the quoted rate is compounded each year

- ***Continuous compounding*** is a compounding that happens all the time, and not at specific intervals. In the case of continuous compounding:

$$EAR = e^{APR} - 1$$

➔ For instance, if we want an annual EAR of 15%, and we want to make it look cheaper, the minimum APR the lenders are required by law to report is:

$0.15 = e^{APR\ min} - 1$ ➔ $APR_{min} = 0.1397 \approx 14\%$

Single period realized return

- The **Holding Period Return** (**HPR**) is the return obtained from holding an asset for a certain period:

$$HPR = \frac{ending\ price + cash\ dividend - beginning\ price}{beginning\ price}$$

➢ The Holding Period Return can be annualized for a period of *t* years:

$$HPR_{annualized} = (1 + HPR)^{1/t} - 1 = \left(\frac{ending\ price + cash\ dividend}{beginning\ price}\right)^{1/t} - 1$$

➔ The annualized holding period return helps us to compare investments with different time horizons

Multiple-periods realized return

➢ *Arithmetic Average*

- It is the arithmetic mean of each period return
- It is not equivalent to the per-period return because it neglects compounding
- It is useful for forecasting the return next period

$$\frac{1}{T} \times (r_1 + r_2 + r_3 + \dots + r_n)$$

➢ *Geometric Average*

- Gives the equivalent per period return, so it considers compounding
- It is useful to evaluate the portfolio performance

$$[(1 + r_1)(1 + r_2)(1 + r_3) + \dots (1 + r_n)]^{1/T} - 1 = \left[\frac{Accumulated\ Value\ (t = T)}{Value\ (t = 0)}\right]^{\frac{1}{T}} - 1$$

➢ *Internal Rate of Return (IRR)*

- Return if one can reinvest profit at this rate
- It is a "*dollar weighted average*"

➔ The IRR is the discount rate that makes the initial price equal to the present value of future cash flows

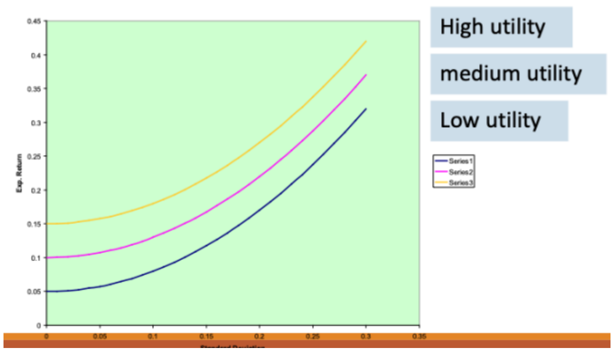
$$P_0 = \sum_{t=1}^{\infty} \frac{C(t)}{(1 + IRR)^t}$$

Risk and Expected Return

- Given two portfolios, an investor will always choose the one with higher expected return and lower risk
  - Investors prefer more to less➔ Prefer a higher Expected Return
  - Investors are risk-averse➔ Prefer a lower standard deviation

➔ Investors must optimally tradeoff risk and return in order to maximize their **expected utility**

- *Indifference Curves* are a set of $(E(R_p), \sigma_p)$ combinations that give an investor the same expected utility



➔ The curves show that, as the risk increases, the required expected return to have the same utility must increase
  - *Risk Averse* investors have very steep-upward sloping utility curves
  - *Risk Neutral* investors have flat utility curves
  - *Risk Loving* investors have downward sloping utility curves

Expected Utility

- Considering the *utility function* of an investor, we can substitute in the equation the Expected Return and the Standard Deviation in order to obtain *utility scores* for each investment and assess which one is the best

$$U = E(R) - A \times \sigma^2$$

➔ In this case, A is a measure of risk aversion:
  - A=0→ Risk-neutral investor
  - A>0→ Risk-averse investor
  - A<0→ Risk-seeking investor

- When an investor is *Risk-Neutral*, A=0 and the investor will always choose the portfolio with the highest expected return, regardless of the level of risk
  - The utility curve will be a flat line

- *Von Neumann-Morgenstern* developed 4 axioms about rational investors and expected utility theory:
  - ***Completeness***
    For two lotteries, A and B, investors either prefer A over B, or B over A, or are indifferent between the two

  - ***Transitivity***
    For three lotteries A, B, and C, if an investor prefers A over B, and B over C, then he will certainly prefer A over C

  - ***Convexity***
    For three lotteries A, B, and C, if the investor prefers A over B, and B over C, then there must be a linear combination of A and C such that the investor is indifferent between B and the combination of A and C with weights *w*:
    $$wA + (1 - w)C$$

  - ***Independence***
    For three lotteries A, B, and C, if A is preferred to B, then a combination of A and C is preferred over the same combination of B and C

- *Power Utility Function* ➔ $U(R) = \frac{\delta R^{1-\gamma}}{(1-\gamma)}$

  - Assigns a score to the return of the portfolio
  - Different scores are assigned by changing R
  - γ is a measure of risk aversion
    → a high positive γ means risk aversion, while $\gamma = 0$ means risk neutrality
  - δ is a time preference parameter, and specifies how an investor prefers consumption today relative to the future

➔ We are able to change *R* to assign a score to different portfolio and maximize the utility

- *Exponential Utility* ➔ $U(R) = \frac{1-e^{-a \times R}}{a}$ for *a* different form 0
  - In this case, *a* is a risk aversion parameter

- If an investor can choose between a risky asset with return R and a risk-free asset with return R_f, he can maximize his expected utility by changing the weight of each asset in the portfolio
  $$max_{\ \omega} E[U(\omega R + (1 - \omega) R_f]$$

  - Since the measure involves expectations, we need a measure of the distribution for the portfolio to evaluate for which ω utility is maximized

Portfolio Selection with Two Risky Securities

- The expected return of a portfolio is:

$$E(R_p) = \sum_{i=1}^{N} \omega_i \times E(R_i)$$

- With two securities, the portfolio variance and standard deviation are:

$$\sigma_p^2 = \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\rho_{1,2}\sigma_1\sigma_2 \Rightarrow \sigma_p = \sqrt{\sigma_p^2}$$

➔ Recall that the *volatility* of a portfolio is its standard deviation

- Given the Assets JP and US, such that:

	mean	volatility
US	13.6%	15.4%
JP	15.0%	23.0%

and with correlation 27%.

And $\omega_{\text{JP}} = 40\%$ and $\omega_{\text{US}} = 60\%$

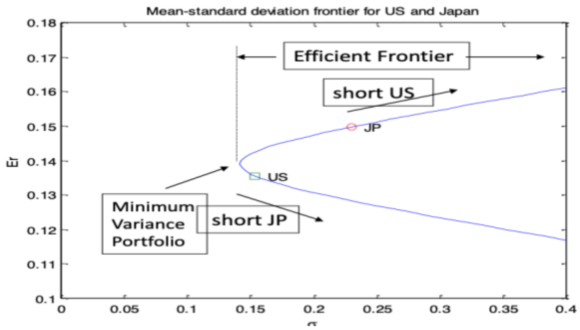
$E\left(R_p\right) = 0.6 \times 0.136 + 0.4 \times 0.15 = 0.142 = \mathbf{14.2}\%$

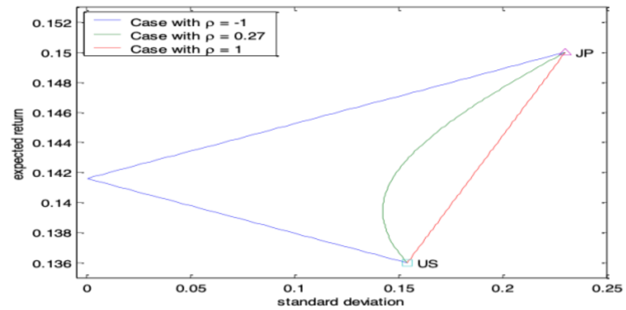
$$\sigma_p^2 = 0.6^2 \times 0.154^2 + 0.4^2 \times 0.23^2 + 2 \times 0.6 \times 0.4 \times 0.27 \times 0.154 \times 0.23 = 0.022$$

$$\sigma_p = 0.147 = \mathbf{14.7}\%$$

➔ Looking at the US market alone, it has a lower expected return and higher volatility than this diversified portfolio

- If we vary the weights assigned to each asset in a portfolio, we obtain the *Investment Opportunity Set*
  - The *investment opportunity* set consists of all available risk-return combinations
  - An *efficient portfolio* is a portfolio that has the highest possible expected return for a given standard deviation
  - The *efficient frontier* is the set of efficient portfolios
    ➔ It is the upper portion of the minimum variance frontier starting at the minimum variance portfolio
  - The *minimum variance portfolio* (**mvp**) is the portfolio that provides the lowest volatility among all possible portfolios of risky assets





➔ All portfolios above the Minimum Variance Portfolio are on the efficient frontier, however the ones below the Minimum Variance Portfolio are called inefficient portfolio because their expected return decreases as volatility increases

➢ In order to choose the optimal portfolio, we can calculate some indifference curves for a given level of utility:
  - By setting a utility level, we can obtain all the different combinations of Expected returns and Volatility that yield the same utility level
  - The more the utility curve is top-left, the more the utility is maximized
  - The utility curve that represents the optimal portfolio is tangent to the Investment Opportunity Set
  - The lower the risk aversion coefficient, the riskier the optimal portfolio will be. Indeed, only if an investor was infinitely risk-averse he would choose as the optimal portfolio the minimum variance portfolio

➢ Correlation between two assets also has an impact on diversification:
  - Perfectly positive correlation leads to the complete annulment of diversification
  - Perfectly negative correlation leads to the selection of a portfolio with zero risk and some level of expected return
  - If the portfolio with perfectly negatively correlated assets has a positive expected return, then the expectation on this portfolio must be identical to the risk-free asset

➢ Any *mean-variance* investor should choose an efficient portfolio to benefit from diversification:
  - The specific choice depends on the investor's risk aversion
  - A more risk-averse individual should choose a portfolio with lower risk and lower expected return

Portfolio Selection with a Riskless Security

- The risk-free return is denoted R_f and its return is known for sure:

  $$E(R_f) = R_f \; ; \; \sigma_f^2 = 0 \; ; Cov(R_f, R_i) = 0 \; for \; any \; other \; asset \; i$$

- Let ω be the fraction of wealth invested in the risky asset, and $(1-\omega)$ the fraction of wealth invested in the risk-free asset

  - The expected return of the portfolio is:
    $E(R_p) = \omega \times E(R_i) + (1-\omega)R_f = R_f + \omega \times E(R_i - R_f)$
    $E(R_i - R_f)$ is called the excess return on the risk-free asset

  - The Variance of the portfolio is just the variance of the risky asset times the weight on the asset
    $\sigma_p^2 = \omega^2 \times \sigma_i^2$

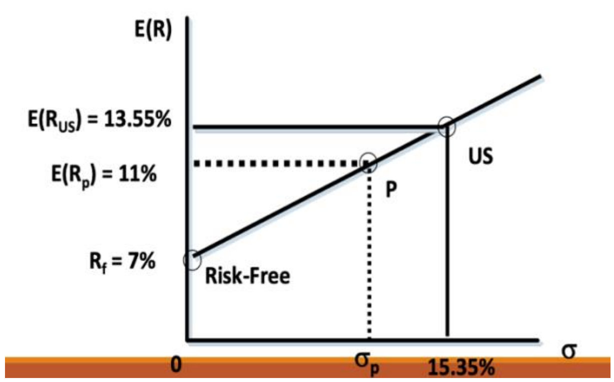
  - The standard deviation of the portfolio is:
    $\sigma_p = |\omega| \times \sigma_i$

  - There is no covariance because the risk-free asset is not correlated with any other asset

- Considering various portfolios *p*, which are long in the risky asset and either long or short in the riskless asset, the **risk-return relationship** is:

$$E[R_p] = R_f + \frac{E[R_i] - R_f}{\sigma_i}\sigma_p = R_f + (Sharpe \; ratio \; of \; i) \times \sigma_p = \boldsymbol{R_f + SR_i \sigma_p}$$

➔ Which is known as the **capital allocation line**



- If volatility is zero, then the portfolio earns the risk-free rate
- As portfolio volatility increases, the Expected return increases by the Sharpe Ratio for unit of volatility
- The Sharpe Ratio is the slope of the Capital Allocation Line

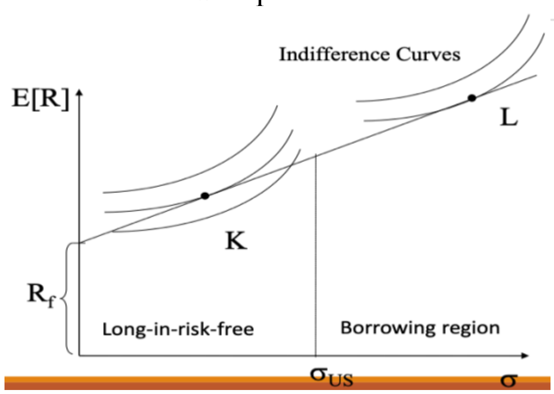
➢ The *Sharpe Ratio* indicate the return premium per unit of volatility

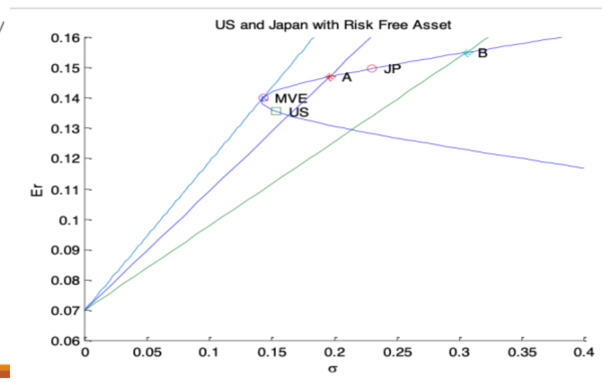
$$Sharpe\ Ratio = \frac{E[R_i] - R_f}{\sigma_i}$$

➢ When the portfolio volatility is zero, it means that we have invested everything in the riskless asset:

- The Sharpe Ratio indicates how much return in excess of the risk-free rate we have earned by investing in the risky asset instead of the risk-free asset

➢ Using the Capital Allocation Line, the Optimal Portfolio Choice is the portfolio that maximizes the Sharpe Ratio





- We are looking for the portfolio of risky and riskless assets for which the slope of the Capital Allocation Line is as high as possible
- This means that we are looking for the portfolio with the highest expected return of the risky asset and lower portfolio variance

➢ Steps to create an Optimal Portfolio Selection:

- Create the set of mean-St Dev. combinations from different portfolios of risky assets
- Find the tangency portfolio, which is the portfolio with the **highest Sharpe Ratio**
- Choose the combination of the tangency portfolio and the risk-free asset to suit your risk-return preferences

➢ An investor's risk aversion determines the fraction of wealth invested in the risk-free asset, but all investors should have the rest of their wealth invested in the tangency portfolio

Portfolio choice with higher-moment risk preferences

- Recall that for two assets, one risky with a gross return and one risk-free with a return of R_f, the portfolio return is:

$$R_p = \omega R + (1-\omega)R_f$$

- And the expected return is:

$$E(R_p) = \omega E(R) + (1-\omega)R_f$$

Because $E(R_f) = R_f$

- The power utility function is a common representation of preferences:

$$U(R) = \frac{\delta c^{1-\gamma}}{(1-\gamma)}$$

  - Where *c* is consumption, δ is a time-preference parameter, and γ is risk-aversion

- In portfolio theory, we often proxy consumption by the return on the portfolio:

$$U(R_p) = \frac{{R_p}^{1-\gamma}}{(1-\gamma)}$$

  - In this case, δ is removed for simplicity, because it only matters when there are intertemporal choices

- The optimization problem for an investor is maximizing his **expected utility** by moving around ω and investing more or less in the risk-free asset:

$$max_{\omega}\ E[U(\omega R + (1-\omega)R_f]$$

- In order to make progress, we need information on the distribution of portfolio returns:
  - If we knew the entire distribution, then we could simply evaluate the expectation for each value of ω
  - However, we usually know the **specific moments** of the distribution, which can be used to *approximate* the expected utility of the portfolio

- One common way to express the utility function is in terms of a **Taylor Expansion**:
  - A third order Taylor expansion of the utility function around an initial guess (a) is:

$$U(R_p) \approx U(a) + U'(a) \times (R_p - a) + \frac{U''(a)}{2} \times (R_p - a)^2 + \frac{U'''(a)}{6} \times (R_p - a)^3$$

  - By adding more terms to the expansion, we can make the approximation better

➢ By choosing ***a*=*1*** and $U(R_p) = \frac{{R_p}^{1-\gamma}}{(1-\gamma)}$, we have:

$$U(1) = \frac{1}{1-\gamma};\ U'(1) = 1;\ \ U''(1) = -\gamma$$

And the approximate utility function is:

$$U(R_p) \approx \frac{1}{1-\gamma} + (R_p - 1) - \frac{\gamma}{2}(R_p - 1)^2$$

➔ Therefore, the expected utility is:

$$max_\omega\ E\left(\frac{1}{1-\gamma} + (R_p - 1) - \frac{\gamma}{2}(R_p - 1)^2\right)$$

Which is the same as maximizing:

$$max_\omega\ E(R_p) - \frac{\gamma}{2}E\left((R_p - 1)^2\right)$$

- In the maximization of the expectations, we can get rid of all the constants because they only increase or decrease the level of utility
- Constants do NOT matter for the *optimal choice of ω*

➔ Using the approximation:

$$E\left((R_p - 1)^2\right) \approx E\left((R_p - E(R_p))^2\right) = var(R_p)$$

We can write:

$$max_\omega\ E(R_p) - \frac{\gamma}{2}var(R_p)$$

Which is very similar to the utility function previously used:

$$U(R_p) = E(R_p) - \frac{1}{2} \times A \times \sigma^2$$

➢ If, however, we choose ***a*= $\boldsymbol{E(R_p)}$** and make a third order approximation of the utility function, we have:

- $U(E(R_p)) = \frac{E(R_p)^{1-\gamma}}{1-\gamma}$
- $U'(E(R_p)) = E(R_p)^{-\gamma}$
- $U''(E(R_p)) = -\gamma E(R_p)^{-\gamma-1}$
- $U'''(E(R_p)) = \gamma(\gamma+1)E(R_p)^{-\gamma-2}$

➔ Therefore, the approximate utility function will be:

$$U(R_p) \approx \frac{\mathrm{E}(R_p)^{1-\gamma}}{1-\gamma} + \mathrm{E}(R_p)^{-\gamma}\left(R_p - \mathrm{E}(R_p)\right) - \frac{\gamma \mathrm{E}(R_p)^{-\gamma-1}}{2}\left(R_p - \mathrm{E}(R_p)\right)^2 + \frac{\gamma(\gamma+1)\mathrm{E}(R_p)^{-\gamma-2}}{6}\left(R_p - \mathrm{E}(R_p)\right)^3$$

Which is similar to optimizing:

$$\max_{\omega} \frac{\mathrm{E}(R_p)^{1-\gamma}}{1-\gamma} - \frac{\gamma \mathrm{E}(R_p)^{-\gamma-1}}{2} var(R_p) + \frac{\gamma(\gamma+1)\mathrm{E}(R_p)^{-\gamma-2} var(R_p)^{\frac{3}{2}}}{6} skewness(R_p)$$

Indeed, recall that:

$$skewness(R_p) = \frac{E[(R_p - E(R_p))^3]}{var(R_p)^{3/2}}$$

- From the maximization, we can see that:
  - Higher $E(R_p)$ leads to a higher utility
  - Higher $var(R_p)$ leads to lower utility
  - Higher $skewness(R_p)$leads to a higher utility

- Therefore, we can identify **three moments** of the portfolio:
  - *Expected Return*

    $$E(R_p) = \omega E(R) + (1-\omega)R_f$$

  - *Variance*

    $$var(R_p) = \omega^2 var(R)$$

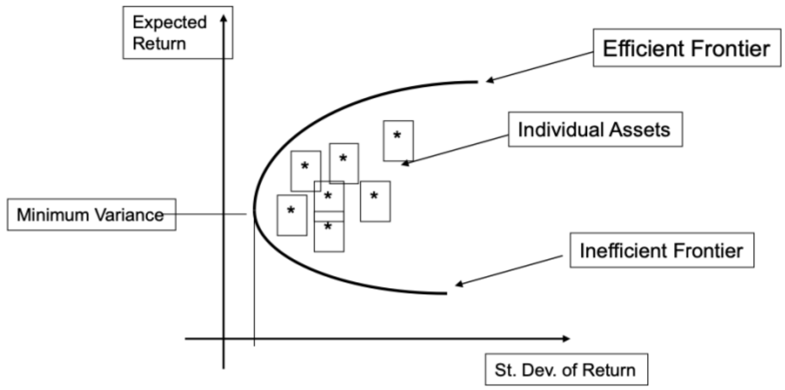
  - *Skewness*

    $$skewness(R_p) = \frac{E[(R_p - E(R_p))^3]}{var(R_p)^{3/2}} = skewness(R)$$

➔ Therefore, knowing the expected return, the variance, and the skewness of R, as well as the risk-free return, then we can optimize the expected utility

Portfolio Selection with Multiple Risky Securities

- The investment opportunity set with many assets can be represented as:



- Firstly, we want to carry out the optimal portfolio selection with many risky assets and a risk-free asset:

    1. Create a set of the possible mean-standard deviation combinations from different portfolios of risky assets and find the efficient frontier of the risky assets portfolio
    2. Find the *tangency portfolio*, that is, the portfolio with the highest Sharpe Ratio
    3. Choose the combination of the tangency portfolio and the risk-free asset to suit your risk-return preferences

- Therefore we can talk about the *Two-Fund Separation*, assuming all investors hold combinations of the same two mutual funds:
    - The risk-free asset
    - The tangency portfolio

➔ An investor's risk aversion determines the fraction of wealth invested in the risk-free asset; however, all investors should have the rest of their wealth invested in the tangency portfolio

- Portfolio Diversification helps to reduce risk:
    - Suppose that we start with a typical US stock and add equally weighted stock to the portfolio

➔ If we have an equally weighted portfolio with weights 1/N on N independent stocks, then the variance of the portfolio return is:

$$\sigma_p^2 = \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^2 = \frac{1}{N} * (Average\ Variance)$$

- Therefore, as the number of assets increases, the risk diversified away due to the **insurance principle**
- As N approaches infinity, σ_p^2 approaches zero

- In the real world, however, assets are actually correlated:

$$\sigma_p^2 = \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^2 + \frac{2}{N^2}\sum_{i=1}^{N}\sum_{j>1}^{N} Cov(R_i, R_j) =$$
$$= \frac{1}{N} * (Average\ Variance) + (1 - \frac{1}{N})(Average\ Covariance)$$

➔ In this case, as N goes to infinity, the risk of the portfolio becomes equal to the Average Covariance, which is called the ***non-diversifiable risk*** of a portfolio
  - The non-diversifiable risk can be reduced by pooling together a portfolio with negatively correlated assets
  - Assuming that two assets are positively correlated, as the correlation between the two assets increases, the proportion of the asset with lowest expected return is reduced
  - Non-diversifiable risk can also be called *covariance risk* or *systematic risk*
  - Some examples are market risk, macroeconomic risk, or industry risk

- The part of risk that can be diversified away in a large portfolio is called:
  - *Idiosyncratic risk, non-systematic risk, diversifiable risk, or unique risk*
  - This kind of risk can be diversified by pooling assets in a large, diversified portfolio

- Therefore, when held in a portfolio some of the risk of a stock disappears, or the risk contribution of the stock to the portfolio is less than the risk if held in isolation:

$$\text{Total Risk in a Stock} = [\text{Non} - \text{Diversifiable Risk}] + [\text{Diversifiable Risk}]$$

The tangent portfolio with two risky assets

- Assume that an investor faces the problem of finding the optimal risky portfolio, he has to choose between two risky assets A and B:
  - A has *excess return* $R_A = r_A - r_f$ and variance $= \sigma_A^2$
  - B has *excess return* $R_B = r_B - r_f$ and variance $= \sigma_B^2$

➔ The tangency portfolio will be the one that maximizes the Sharpe Ratio

- Therefore, the investor maximization problem will be in this case:

$$max_\omega\ S_p = \frac{E(R_p)}{\sigma_p}$$

Under the constraint that $\omega_B = 1 - \omega_A$

➔ To solve the maximization problem, find the First Order Condition, set it equal to zero, and solve for ω_A

- Hence, the solution to the investor's maximization problem for two risky assets will be:

$$\omega_A = \frac{E(R_A)\sigma_B^2 - E(R_B)cov(R_A, R_B)}{E(R_A)\sigma_B^2 + E(R_B)\sigma_A^2 - \big(E(R_A) + E(R_B)\big)cov(R_A, R_B)}$$

And ω_B will simply be equal to $1 - \omega_A$

Index Factor Models

- The Markowitz model is a great theoretical model which, however, faces some empirical challenges:
  - Firstly, the quality of input parameters must be high
  - Secondly, the number of estimates needed increases rapidly with the number of stock, and bad estimates can lead to bad portfolio allocation decisions

- In order to simplify the model, we can use *Single Factor models*, which simplify the economy such that an asset's return is characterized by:
  - *Expected return*
  - A *systematic risk factor* common for all assets (**m**)
  - A *risk component specific* to a certain asset (**e**$_\mathbf{i}$)

- Therefore, the *return* for an asset *i* will be:

$$r_i = E(r_i) + \beta_i m + e_i$$

<u>*Assumptions*</u>

  - $\boldsymbol{cov(m, e_i) = 0}$
  - $\boldsymbol{cov(e_j, e_i) = 0}$ for all other assets *j*
  - E(m)=E(e_i)=E(e_j)

- The *variance* for each asset *i* will be:

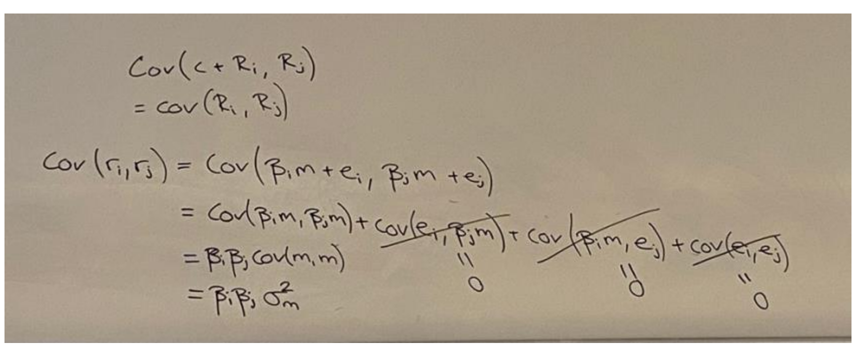
$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{e_i}^2$$

since $cov(m, e_i) = 0$

- The *Covariance* between asset *i* and asset *j* will be:

$$Cov(r_i, r_j) = Cov(\beta_i m + e_i, \beta_j m + e_j) = \boldsymbol{\beta_i \beta_j \sigma_m^2}$$

since $cov(e_j, e_i) = 0$



- Therefore, using the Index Factor Models, we can estimate a portfolio made of 50 stocks using:
    - 50 estimates of expected returns ➔ hardest thing to estimate
    - 50 estimates of firm specific variances ➔ can be estimated using past returns
    - 1 estimate of factor variance ➔ can be estimated using past returns
    - 50 estimates of *factor loadings*

- One possible factor can be the aggregate stock market, also called the *Index Model*
    - Set $m = R_m$, where $R_m = r_m - r_f$ is the market's excess return
    - Write the excess return on asset *i* as:

$$R_i = \alpha_i + \beta_i R_m + e_i$$

α_i is the excess return when market excess return is zero
β_i is the loading on the risk factor

➔ We can estimate α_i and β_i by running a time-series regression:

$$R_i(t) = \alpha_i + \beta_i R_m(t) + e_i(t)$$

- Suppose we choose an equal weight portfolio of n securities
    - The excess rate of return on each asset is:

$$R_i = \alpha_i + \beta_i R_m + e_i$$

    - Where the portfolio variance is:

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sigma_{e_p}^2$$

Notice that $\sigma_{e_p}^2$ is the idiosyncratic variance, which diminishes as n→∞

$$\sigma_{e_p}^2 = Var\left(\frac{1}{n}\sum_{i=1}^{n} e_i\right) = \frac{1}{n}\bar{\sigma}_e^2$$

Where $\bar{\sigma}_e^2$ is the average of the firm specific variances

The Capital Asset Pricing Model

- The CAPM is an equilibrium model that:
  - Predicts optimal portfolio choices
  - Predicts the relationship between risk and expected return
  - Underlies much of modern finance theory
  - Underlies most of real-world financial decision making

- The model is derived using Markowitz's principles of portfolio theory, with additional simplifying assumptions
  - Sharpe, Lintner, and Mossin are the researchers credited with its development

- *Assumptions*

1. *The market is in a competitive equilibrium*
   - Supply is fixed and it equals demand
   - If demand is greater than supply, the excess demand drives up prices
   - Investors are price-takers (take prices as given)
   - No investor can manipulate the market and no monopoly is allowed

2. *Single-period investment horizon*
   - All investors agree on a certain time horizon
   - This ensures that all investors are facing the same investment problem

3. *All assets are tradable*
   - This ensures that every investor has the same assets to invest in
   - All the assets in the world are called the *market portfolio*

4. *No frictions*
   - No taxes and no transaction costs (no bid-ask spread)
   - Same interest rate for lending and borrowing
   - All investors can borrow or lend unlimited amount, so there are no margin requirements

5. *Investors are rational mean-variance optimizers with homogenous expectations*
   - Investors choose efficient portfolios consistent with their risk-return preferences
   - Investors have the same views about expected return, variances, and covariances (and hence correlations)

- Some of the above assumptions can be relaxed, and the CAPM will still hold
  - Indeed, the model is a good approximation of reality in any case
  - If many assumptions are relaxed, generalized versions of CAPM applies

➢ From portfolio theory, recall that:
- All investors should have a fraction of their wealth invested in the risk-free security
- The rest of their wealth is invested in the tangency portfolio
- The tangency portfolio is the same for all investors

➔ Therefore, since in equilibrium **supply=demand**, the *tangency portfolio* must be the portfolio of *all existing risky assets*, the so-called **market portfolio**

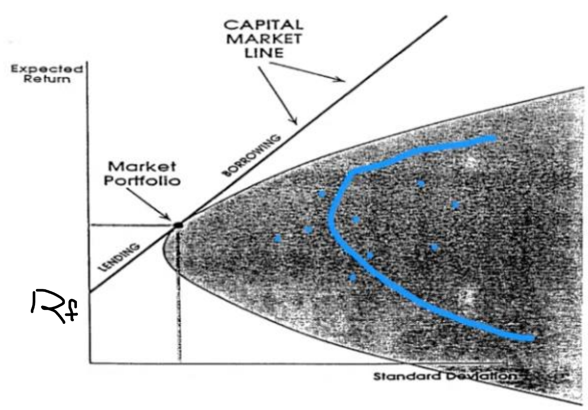
➢ The **market portfolio** is the portfolio consisting of all assets
- However, an investor can also invest in the market portfolio if he buys a share of every security, *weighted by their market cap*

$$\omega_{iM} = \frac{p_i \times n_i}{\sum p_i \times n_i} = \frac{market\ capitalization\ of\ security\ i}{total\ market\ capitalization}$$

➢ The market portfolio's *Capital Allocation Line* is called the **Capital Market Line** in the Capital Asset Pricing Model
- The Capital Market Line gives the *risk-return combinations* achieved by forming portfolios from the risk-free security and the market portfolio:

$$E(R_p) = R_f + \frac{[E(R_M) - R_f]}{\sigma_M}\sigma_p$$

- Recall that the Capital Allocation Line with the highest Sharpe Ratio is the Capital Asset Line with respect to the tangency portfolio
- In equilibrium, the market portfolio is the tangency portfolio



➢ The model shows that investor should invest part of their wealth in the Market Portfolio
- The rest of their wealth should be invested in the risk-free asset
- The **Capital Market Line** shows the different combinations of Market Portfolio and risk-free asset that provide different expected return and standard deviation

- If the investor is in the borrowing region, he must have shorted the risk-free asset to invest more in the market portfolio
  - If, however, the investor is more risk-averse, and is in the lending region, he must have invested only part of his wealth in the Market Portfolio, and put the remainder in the risk-free asset

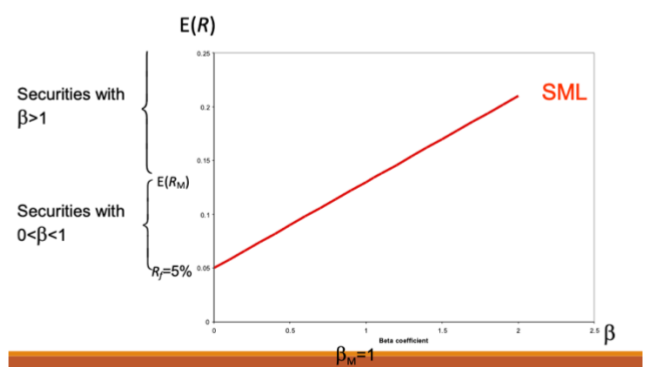
- The CAPM is most famous for its prediction concerning the relationship between risk and return of individual securities:

$$E(R_i) = R_f + \beta_i \times [E(R_M) - R_f]$$

Where $\boldsymbol{\beta_i} = \frac{\mathbf{cov(R_i, R_M)}}{\boldsymbol{\sigma^2_M}}$, which measures the security's sensitivity to market movements

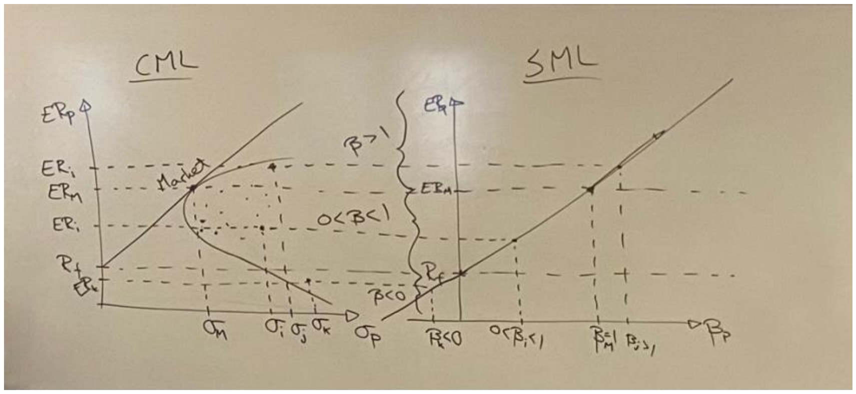
  - The model predicts that the expected return of an asset is linear in its *beta*
  - This linear relation is called the **Security Market Line** (**SML**)
  - Notice that the Excess Return of an asset is always larger than zero, which implies that the expected return on the market portfolio cannot be less than the risk-free rate

- The **Security Market Line** plots the expected return of an asset or a portfolio according to the beta:
  - The expected return should increase in a directly proportional way to the beta of the asset



→ So keep in mind:
  - If we describe the expected return of a portfolio using its volatility, we obtain the **Capital Market Line**
  - If we describe the expected return of a portfolio using its beta, we obtain the **Security Market Line**

➢ The *Expected Return - Standard Deviation Frontier* and the Betas can be plotted together using a graph that relates the location of the individual securities with respect to the Mean-Standard Deviation Frontier to their betas



Systematic and Non-Systematic Risk

➢ The CAPM can be written as:

$$\mathrm{R_i = R_f + \beta_i(R_m - R_f) + error_i}$$

Where:

- $\boldsymbol{\beta_i} = \frac{\mathbf{cov(R_i,R_M)}}{\boldsymbol{\sigma^2_M}}$
- $\mathrm{E(error_i) = 0}$
- $\mathrm{Cov(error_i, R_M) = 0}$

➔ This implies that the total risk of a security can be partitioned into two components:

$$\underbrace{\sigma_i^2}_{\substack{var(R_i)\\ total\\ risk}} = \underbrace{\beta_i^2\sigma_M^2}_{\substack{market\\ risk}} + \underbrace{\bar{\sigma}_i^2}_{\substack{var(error_i)\\ idiosyncra\ tic\\ risk}}$$

➢ $\boldsymbol{\beta_i}$ measures the contribution to the total risk of a well-diversified portfolio, namely the market portfolio

- Hence, it measures the non-diversifiable risk of the stock
- Investors must be compensated for holding non-diversifiable risk, and this explains why the CAPM predicts that Expected Return increases with Beta

$$\mathrm{E(R_i) = R_f + \beta_i \times [E(R_M) - R_f]}$$

- The size of the compensation depends on the equilibrium risk premium $[E(R_M) - R_f]$, which is increasing in the variance of the market portfolio, and the degree of risk aversion of the average investor

Estimating Beta

- Many institutions give estimates for stocks Betas:
  - Bloomberg
  - Yahoo
  - Merrill Lynch

- The stock beta can also be estimated using **Linear Regressions**
  - Start with the CAPM equation

$$E(R_i) = R_f + \beta_i \times [E(R_M) - R_f]$$

  - Get data on the excess returns of the individual stock and the market:

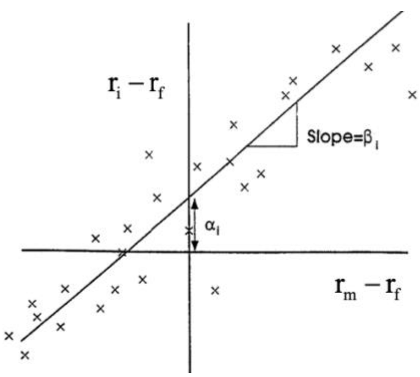
$R_M^e(t) = R_m(t) - R_f$ $\qquad\qquad$ $R_i^e(t) = R_i(t) - R_f$

  - The beta can be estimated by running the following regression, typically using at least 60 months of data:

$$R_i^e(t) = \alpha_i + \beta_i R_M^e(t) + error_i(t)$$

  - The **Security Characteristic Line** is the regression line:

$$R_i^e(t) = \alpha_i + \beta_i R_M^e(t) + error_i(t)$$



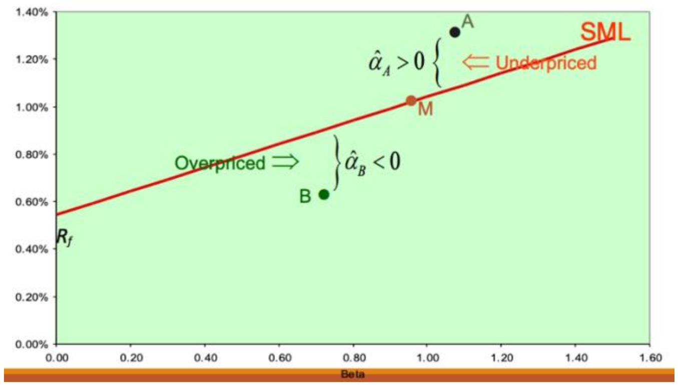
  - Note that CAPM implies that $\alpha_i = 0$, because the excess return on the asset is equal to the excess return on the market times the beta. No intercept is contemplated

- The CAPM model has many different applications
  - Portfolio Choice
  - Shows what a fair security return should be
  - Gives a benchmark for security analysis
  - It is used to estimate the required return in capital budgeting to compute NPV of risky projects
  - It is used to evaluate the performance of fund manager

- One possible benchmark for stock selection is to find assets that are cheap relative to the CAPM:
    - A security's alpha is defined as

$$\alpha_i = E[R_i] - R_f - \beta_i \times [\mathrm{E}(\mathrm{R_M}) - \mathrm{R_f}]$$

    - Some fund managers try to buy positive-alpha stocks and sell negative-alpha stocks
    - This is because the CAPM predicts that alpha should be zero



    - Stock B in the picture above has a negative alpha since its expected return is too low for a given beta
      ➔ Therefore, the stock is overpriced, and investors should sell it

    - Stock A has a positive beta since its expected return is higher than the CAPM predicts for a given beta
      ➔ Therefore, investors should buy stock A since it is underpriced

➔ As investors exploit the investment opportunity, stocks will eventually provide the expected return predicted by the Security Market Line

- An *active strategy* tries to beat the market by stock picking, timing, or other methods

➔ However, the CAPM implies that:
    - Security analysis is not necessary
    - Every investor should just hold a mix of the risk-free security and the market portfolio, which means pursuing a **passive strategy**

Testing the CAPM and its extensions

- Recall that the CAPM equation is:

$$E(R_i) = R_f + \beta_i \times [E(R_M) - R_f]$$

  - We can run a regression of excess return on asset *i* onto the market risk premium to estimate the asset's beta and alpha:

$$R_i^e(t) = \alpha_i + \beta_i R_M^e(t) + error_i(t)$$

➔ The CAPM implies that alpha equals 0

- Suppose we repeat this process for N assets, giving us a vector of asset specific alphas:
  - An idea for testing the CAPM is to test if the alphas are all jointly zero
  - However, a rejection of the hypothesis that alphas are jointly zero does NOT invalidate the model

- Roll's critique about the CAPM is that the market portfolio is unobservable:
  - Indeed, we might falsify the CAPM because we did not use the true market portfolio in our regressions

- The CAPM is a one period model:
  - It holds period by period
  - Implementing the CAPM through a regression implies that the betas are constant over time, which they might not be
  - Therefore, we must consider *conditional betas*

- In the CAPM the only source of risk is the market portfolio:
  - However, since the market portfolio is unobservable, we usually proxy it by some stock market index
  - If the stock market index does not capture all risk of the market portfolio, then we normally can potentially find other risk factors that can help describe expected asset returns

➔ Therefore, assuming the CAPM holds, we need to find factors that can span the risk of the true market portfolio

Fama-French three factors model (FF3)

- Eugene Fama and Kenneth French observed the following:
  - Small companies tend to outperform big companies
  - Firms with high book-to-market ratios tend to outperform firms with low book-to-market ratios

➔ From these observations, they created the two risk factors, which allows to sort stocks depending on their market capitalization or book-to-market ratios:
  - **SMB** ➔ Small market capitalization Minus Big market capitalization
  - **HML** ➔ High book-to-market Minus Low book-to-market

- Once the factors are created, the FF3 model predicts that the expected return on asset *i* is:

$$\mathbf{E(R_i) - R_f = \beta_i \times [E(R_M) - R_f] + \boldsymbol{\beta_i^{SMB}} E(SMB) + \boldsymbol{\beta_i^{HML}}\, E(HML)}$$

  - $\boldsymbol{\beta_i^{SMB}}$ and $\boldsymbol{\beta_i^{HML}}$ are the assets sensitivity to the *new* risk factors, which can be estimated using a linear regression

Conditional CAPM

- Another possible problem with the regression approach is that we estimate constant betas over time
  - The CAPM is a one-period model, and betas might change over time

- Some possible solutions to these problems are:
  - Using short horizon rolling window regressions, even though estimating short horizon correlations is very noisy
  - Use instruments to make betas time-varying, and that they depend on some instrument which is time varying. The main issue with this approach is that it is difficult to choose which instrument to use.

CAPM in Academia vs CAPM in the industry

- In Academia, we still search for the right specification of the CAPM
  - We try to incorporate the best possible CAPM
  - We use multifactor models to evaluate new risk factors and investor performance
  - Usually, new models are evaluated using the FF3 or the FF5, which includes both profitability and investments are risk

- In the real world, we are primarily concerned with the standard specification of the CAPM, since beating the US stock market is already difficult:
  - Active investors want to beat the stock market, but on average they do not manage to do so

Exercise

Are the following true or false?

- Stocks with a beta of zero offer an expected return of zero
- The CAPM implies that investors require higher return to hold highly volatile securities
- You can construct a portfolio with a beta of 0.75 by investing 0.75 of the investment budget in T-bills and the remainder in the market portfolio

1- False, beta zero implies that stock offers an expected return equal to the risk-free rate
2- False, the CAPM implies that higher market sensitivity makes investors require higher returns. The model is not concerned with idiosyncratic risk.
3- False, the beta of the portfolio is equal to the beta of the fraction of the portfolio invested on the market portfolio

Equity Valuation

- The *Book Value* considers the historical values of an asset, which is the cost of acquisition
- The *Market Price* considers the current value of assets and liabilities
- The *Liquidation Value* is the lower bound on market value:
    - It is the value a firm can be sold for instantaneously by selling or scrapping all of its assets
    - If the market value is lower than the liquidation value, it will be a takeover target since the buyer can buy it for the market value and liquidate it for the higher liquidation value

- The *Replacement Cost* is the amount of money required to replace an existing asset with an equally valued or similar asset at the current market price
    - if the market value is far above the replacement cost, competitors are likely to move into the business
    - The resulting competitive pressure would drive down profits and the market value of all firms until they fell to replacement cost

- The **fundamental value** (**intrinsic value**) is the true value of an asset:
    - First, we find the required rate of return E(R) using the CAPM
    - Then, we set it equal to:

$$E(R) = \frac{E(D_{t+1}) + E(V_{t+1})}{V_t} - 1$$

➔ Therefore, we can express V_t as a function of V_{t+1} and D_{t+1}:

$$V_t = \frac{E(D_{t+1}) + E(V_{t+1})}{1 + E(R)}$$

Dividend Discount Model

- The DDM is derived by using the fundamental-value equation repeatedly:

$$V_0 = \sum_{t=1}^{t} \frac{E(D_t)}{(1 + E(R))^t}$$

Gordon's Growth Model

- If we assume that the expected dividends grow at a constant rate *g*, then by the dividend discount model we get:

$$V_0 = \frac{D_0(1+g)}{1+E(r)} + \frac{D_0(1+g)^2}{(1+E(r))^2} + \frac{D_0(1+g)^3}{(1+E(r))^3} + \cdots = \frac{D_0(1+g)}{E(R)-g} = \frac{\boldsymbol{D_1}}{\boldsymbol{E(R)-g}}$$

- The main assumption is that the Expected Return is higher than the growth rate
- If dividends were to grow forever at a rate faster than k, the value of the stock would be infinite
- Hence, if analysts derive g>k, the rate must be unsustainable in the long-run

Valuation Ratios and Relative Valuation

- **Price-Dividend ratio**:
  - Assuming that $V_0 = P_0$ and that the Gordon's Growth Model applies:

$$\frac{P_0}{V_0} = \frac{1+g}{E(R)-g}$$

- **Price-Earnings ratio**:
  - Assuming that the Gordon's Growth model applies:

$$\frac{P_0}{E_0} = \frac{(1+g)(1-b)}{E(R)-g}$$

  Where *b* is the **retention ratio** (or **plowback ratio**), that is the ration of earnings not distributed as dividends:

$$D_0 = (1-b)E_0$$

- **Price-to-Book ratio**
  - Ratio of market price over the book value of its assets and liabilities

Two-Stages DDM

- A company can grow exceptionally for a while, but at some point, the company matures, and its growth normalizes
  - Suppose that a company's growth rate will reach its long-run level of *g* after 3 years:

$$P_0 = \frac{E(D_1)}{1+E(R)} + \frac{E(D_2)}{(1+E(R))^2} + \frac{E(D_3)}{(1+E(R))^3} + \frac{E(P_3)}{(1+E(R))^3}$$

  Where:

$$P_3 = \frac{(1+g)D_3}{E(R)-g}$$

Arbitrage

- In finance theory, an arbitrage is defined as:
  - A trading strategy that generates a completely riskless profit
  - A trading strategy that generates a positive cash flow at some time and non-negative cash flows at all times

- On Wall Street, arbitrage is often referred to as:
  - A trading strategy that is expected to make profit
  - This is also called a *statistical arbitrage*

- In Finance, we assume the so called *No-arbitrage condition*:
  - The condition states that in the economy there cannot be any arbitrage opportunities
  - If there were any, however, arbitrageurs would trade aggressively to exploit the arbitrage

➔ By using this condition alone, we can:
  - Compute restrictions on security prices
  - Compute the price of derivatives

- Therefore, we can say that the *No-Arbitrage* condition has important implications:

  1) If two securities have the same payoffs, they must have the same prices

Example 1: Arbitrage Pricing

Suppose there are two kinds of zero-coupon bonds, "CATs" and "TIGRs," both paying a face value of $100 in 1 year.

Suppose that the price of CATs is $98.

What must the price of TIGRs be?

	t=0 price	t=1 payoff
Cats	98	100
Tigers	?	100

- Tigers must cost $98 aswell

How could you make an arbitrage if TIGRs were trading at $97?

- Sell cats buy tigers. Gives $1 today without any costs later

	t=0 cashflow	t=1 payoff
Sell cats	98	-100
buy tigers	-97	100
total	1	0

  - Since CATs and TIGRs are identical assets, they should be both priced at $98
  - If TIGRs were trading at $97, we could exploit the arbitrage opportunity by buying TIGRs and sell CATs
  - Assuming that arbitrageurs had illimited liquidity, they would buy as many TIGRs as possible and sell as many CATs as possible to exploit the arbitrage opportunity

2) If a portfolio has the same payoff as a security, the price of the security must be equal to the price of the portfolio
➔ In this case, the portfolio is called a *replicating portfolio*

Example 2: Arbitrage Pricing with a Replicating Portfolio

Face values of bonds are $100

Table 1	t=0 price	t=1 payoff	t=2 payoff	t=3 payoff
1-year zero	98	100		
2-year zero	96		100	
3year zero	93			100

Suppose
- a zero-coupon bond that matures 1 year from today costs $98
- a zero-coupon bond that matures 2 years from today costs $96
- a zero-coupon bond that matures 3 years from today costs $93

What must be the price of a three year maturity coupon bond with a 10% coupon rate?

Table 2	t=0 price	t=1 payoff	t=2 payoff	t=3 payoff
3-year CB	?	10	10	110

- Idea: buy 1/10 of 1-year ZCB, buy 1/10 of 2-year ZCB, and buy 11/10 of 3-year ZCB -> replicating the payoff of the 3-year CB.

Table 3	t=0 price	t=1 payoff	t=2 payoff	t=3 payoff
1/10 1-year zero	9.8	10		
1/10 2-year zero	9.6		10	
11/10 3-year zero	102.3			110
total	121.7	10	10	110

How could you make an arbitrage if the coupon bond were trading at $120?

Table 3	t=0 price	t=1 payoff	t=2 payoff	t=3 payoff
1/10 1-year zero	9.8	10		
1/10 2-year zero	9.6		10	
11/10 3-year zero	102.3			110
total	121.7	10	10	110

- Idea: sell the replicating portfolio and buy the coupon bond

	t=0 price	t=1 payoff	t=2 payoff	t=3 payoff
sell Replicating portfolio	121.7	-10	-10	-110
buy CB	-120	10	10	110
total	1.7	0	0	0

3) If a self-financing trading strategy has the same final payoff as a security, the price of the security must be equal to the cost of the strategy
➔ This is called a *dynamic hedging strategy*

Example 3: Arbitrage Pricing with Dynamic Hedging

Suppose
- a zero-coupon bond that matures 1 year from today costs $98
- 1 year from today, a zero-coupon bond that matures 2 years from today also costs $98

What must be the price of a zero-coupon bond that matures 2 years from now?

	t=0	t=1	t=2
ZCB_0,1	Price 98	Payoff 100	
ZCB_1,2		Price 98	Payoff 100
ZCB_0,2	?		Payoff 100

- Idea: what is ZCB_1,2 worth at time 0?
  - At t=1 it is worth $98 and at t=0 $98 is worth $98*0.98 = $96.04

How could you make an arbitrage if the bond were trading at $97?
- Idea: sell the two period bond and replicate its cashflow
  - At t=0 buy 0.98 of ZCB_0,1, at t=1 buy ZCB_1,2 -> cost = $96.04

- In the real world, *transaction costs* play a key role:
  - If there are transaction costs, it is more difficult to make an arbitrage trading strategy
  - Therefore, we cannot determine prices exactly using the No-Arbitrage Condition
  - However, we can find an interval in which the price must be, which means that we can find an *upper bound* and a *lower bound* for the price

Example 4: Arbitrage Pricing with Transactions Costs

Suppose there are two zero-coupon bonds, CATs and TIGRs, both paying a face value of $100 in 1 year.

The price of CATs is $98

Supposing the cost of shorting is $1, and the cost of buying is 0, what is, respectively, the highest and lowest possible price of TIGRs?

Table 1	t=0	t=1
buy Cats	-98	100
sell Tigers	99	-100
shorting costs	-1	
total	0	0

Table 2	t=0	t=1
sell Cats	98	-100
buy Tigers	-97	100
shorting costs	-1	
total	0	0

- Price interval of Tigers = [$97,$99]

What if the cost of shorting is $0.50 and the cost if buying is $0.50?

- Price interval is price of cats plus/minus the sum of trading costs -> same as above

Fixed-Income Securities

- The main features of bonds are:
  1) *Issuer*:
     - It may be the US Treasury, the Government, States, municipalities, agencies, foreign governments, or corporations

  2) *Term*, which is the number of years to maturity:
     - Short Term securities mature in less than 1 year, and they are T-Bills, Certificates of Deposit, or Commercial Paper
     - Long Term securities mature in more than 1 year, and they include T-Bonds, Corporate Bonds, or Consols

  3) *Price and Par value*:
     - Par bonds are issued at their face value
     - Discount bonds are issued at a lower price than par value
     - Premium bonds are issued at a price higher than par

  4) *Coupon*:
     - The coupon rate is the annual interest payment per dollar of face value
     - The coupon-period is usually semi-annual
     - Coupon rates can be either fixed or variable (*floater* and *inverse floaters*)
     - Coupons can be either *nominal* or *inflation indexed*
     - Some Bonds do not pay coupons (*zero-coupon bonds*)

  5) *Currency*:
     - Yankee Bonds, Samurai Bonds, Bulldog Bonds, Eurobonds
     - We can also have *gold-denominated* bonds

  6) *Credit Risk*
     - Risk-free bonds
     - Defaultable Bonds➔ Rating bonds assign a rating to bonds according to their default risk

  7) *Seniority and Security*
     - Senior, sub-ordinated, and junior bonds
     - Bonds may be secured by properties and equipment, other assets of the issuer, or income streams
     - Sinking funds provisions (*sinkers*)

  8) *Covenants*
     - These are restrictions on additional issues, dividends, and other corporate actions

9) *Option Provisions*
   - **Callability** gives the issuer the right to pay back the bond before maturity
   - **Putability** gives the bondholder the right to demand payment of the loan before maturity
   - **Convertibility** gives the bondholder the right to exchange the bond for stock of the issuer

➢ For an annual-pay coupon bond, its **yield to maturity** (**YTM**) is the same as the IRR, hence it is the rate that solves the equation:

$$Price = \sum_{t=1}^{T} \frac{Coupon}{(1+YTM)^t} + \frac{Face\ value}{(1+YTM)^T}$$

- If YTM=Coupon Rate ➔ Bond is issued at Par
- If YTM>Coupon Rate ➔ Bond is issued at a Discount
- If YTM<Coupon Rate ➔ Bond is issued at a Premium

➢ For a semi-annual coupon bond, the YTM is computed in 2 steps:

1) Find the semi-annual IRR, which is the rate *r* that solves the equation:

$$Price = \sum_{n=1}^{N} \frac{Coupon}{(1+r)^n} + \frac{Face\ value}{(1+r)^N}$$

2) The YTM is the corresponding annual percentage rate, which is:

$$YTM = APR = 2r$$

$$EAR = (1+r)^2 - 1 = \left(1 + \frac{YTM}{2}\right)^2 - 1$$

➢ Assuming that you buy a bond, then the return on the investment will be equal to the YTM of the bond only if:
- You can re-invest the coupons at the same rate
- You hold the bond until maturity

➔ However, the return will differ from the YTM if:
- Coupons are reinvested at a different rate rather than the YTM
- You sell the bond before maturity at a price that corresponds to a different yield to maturity, since market yields can change

- The holding period return is the solution to the equation:

$$P_0 \times (1 + HPR)^t = V(t)$$

Hence, the Holding Period Return is:

$$HPR = \left(\frac{V(t)}{P_0}\right)^{1/t} - 1$$

Assuming that:

- At time zero, the investor buys the bond at price P_0
- The investor reinvests all dividends until date t
- At time t, the investor sells the bond and the re-invested dividends for a total price of V(t)

- A **forward rate** is an interest rate on a future loan that is fixed today:
  - The forward rate for 1-year lending starting *t* years from now is ***f(t)***
  - The forward rate is determined by arbitrage to be:

$$f(t) = \frac{P_t}{P_{t+1}} - 1$$

Where P_t is the price of a t-years zero-coupon bond

  - Notice that price and interest rates are inversely related, and therefore the forward rate can alternatively be stated as:

$$f(t) = \frac{(1 + YTM_{t+1})^{t+1}}{(1 + YTM_t)^t}$$

Where YTM_t is the yield to maturity on a t period zero

- Forward rates are also traded directly with different financial products, such as:
  - Forward Rate Agreements
  - Eurocurrency Interest Rate Futures
  - Bond Futures

- The collection of Yields to Maturity of zero-coupon bonds has many names:
  - The term structure of zero-coupon bonds yields
  - The term structure of interests
  - The yield curve

➔ The typical shapes of the term structure of interest are:

- Flat
- **Upward sloping** (most typical) or Downward sloping
- Hump shaped

Determinants of the Shape of the Term Structure

Theory 1: Expectations Hypothesis

- The YTM on a long-term bond is determined by the expected future short-term interest rates:
  - The theory basically states that the expected future 1-year interest rate is equal to the forward rate:

$$f(t) = E[r(t)]$$

  - The reason why the yield curve is upward sloping is because investors believe that expected future short-rates will be higher than current short-rates
    ➔ Hence, the price we see now simply reflect the expectations in the market

Theory 2: Liquidity Preference Theory

- Buyers of long-term bonds want to be compensated for:
  - Tying up money for a long time
  - Having a price risk if they need to sell before maturity

➔ Therefore, issuers of bonds are willing to pay a higher interest rate on long term bonds because they can lock in an interest rate for many years:
  - The associated *risk-premium* is known as **liquidity premium**

- Based on the theory, the typical shape of the yield curve is upward sloping because:
  - Issuers are willing to pay more in order to lock in the current interest rate in the long-run
  - Bondholders want to be compensated for extra interest rate risk

Theory 3: Market Segmentation Theory (Preferred Habitat Theory)

- Some investors trade short-term bonds:
  - Hence, their demand and supply determine the short-term interest rates
  - In the same way, other investors trade long-term bonds, thus determining the interest rate with supply and demand

- According to this theory, the yield curve can be either upward sloping, downward sloping, or flat, according to the demand of investors for short-term and long-term bonds

Management of Interest-Rate risk

Interest-Rate Sensitivity

- The *First Order Effect* underlines that bond prices and yields are negatively related

- *Maturity matters* in the interest rate sensitivity:
    - Long-term bonds are more sensitive to changes in the interest rate because cash flows arriving far away in the future are discounted by higher factors if interest rates rise
    - On the other hand, a change in the interest rate has very little impact on the cash flows received in the near future

- *Convexity* states that an increase in a bond's YTM results in a smaller price decline than the price gain associated with a decrease of equal magnitude in the YTM:
    - The longer the maturity, the larger the effect

Duration

- The Duration (D) of a bond with cashflows *c(t)* is defined as minus the **elasticity** of its price (*P*) with respect to 1 plus its yield (y):

$$D = -\frac{dP}{dy} \times \frac{1+y}{P} = \sum_{t=1}^{T} f(t)t$$

  Where $f(t)$ is the fraction of the present value of the bond that comes from $c(t)$:

$$f(t) = \frac{c(t)}{(1+y)'P}$$

➔ We can therefore see that the duration is equal to the average of the cash-flows times *t* weighted by $f(t)$

- The relative price-response to a yield change will therefore be:

$$\frac{\Delta P}{P} \cong -D \times \frac{\Delta y}{1+y} = -\frac{D}{1+y} \times \Delta y = -D^{modified} \times \Delta y$$

- Bonds with cash flows very far in the future will have higher duration, and will therefore be more sensitive to interest rate changes

- A few facts about Duration:
  - The Duration of a portfolio is the weighted average of the durations of its securities, where the weights are the proportions of the portfolio invested in those securities
  - The duration of a ZCB is equal to its maturity, since all the cash flows are deferred until then
  - If a bond pays coupon, its duration is lower than the bond's maturity
  - If the coupon rate increases, keeping everything else equal, the duration of the bond will decrease because a larger fraction of cash flows arrives on the near future
  - The duration of a perpetuity is $\frac{1+y}{y}$

Interest-Rate Management

- Investors and financial institutions are subject to interest-rate risk because a change in the interest rate will lead to:
  - Price risk
  - Re-investment risk

➔ Hence, investors want to construct a portfolio which is insensitive to interest-rate changes

Cash Flow Matching

- This method allows to reduce risk by matching exactly the cash flows of assets and liabilities:
  - In this way, the portfolio will be risk-free and, in particular, it has no interest-rate risk
  - In principle it can be done using ZCB

➔ The main issues with Cash Flow Matching arise because:
1) It can be difficult to find securities that enable a perfect match
2) The needed securities may be illiquid and have high transaction costs

Duration Matching (Immunization)

- If cash flow matching is impossible or very costly, one can instead do duration matching:
  - The process consists in making the duration of assets and liabilities equal
  - Then, this will allow to make the sensitivity to interest-rate changes equal to zero

$$\Delta P = -\frac{D^{assets}}{1+y} P^{assets} \Delta y - \left(-\frac{D^{liabilities}}{1+y} P^{liabilities} \Delta y \right) = 0$$

➔ In this case, interest rate changes will make the values of assets and liabilities change by approximately the same amount

- The main issues with Immunization are that:
  - Portfolios require rebalancing, thus forcing investors to incur transaction costs
  - It is an approximation that assumes a *flat term structure* of interest rates
  - The approximation also assumes only the risk of changes in the interest rate level, but does not consider any variation in the slope of the term-structure or other types of shape changes

Convexity

- The sensitivity of prices with respect to yield is approximated by a linear function when using duration:
  - However, the relationship actually is non-linear, and, in particular, it is **convex**
  - The convexity of a bond is the *curvature* of its *price-yield relationship*:

$$convexity = \frac{d^2P}{dy^2}\frac{1}{P} = \sum_{t=1}^{T} w_t \frac{(t^2 + t)}{(1+y)^2}$$

Where:

$$w_t = \frac{Cashflow(t)}{(1+y)^t P}$$

➔ The relative price response to a yield change can be better approximated using convexity:

$$\frac{\Delta P}{P} \cong -\frac{D}{1+y} \times \Delta y \times \frac{1}{2} convexity \times (\Delta y)^2$$

Options and Derivatives

- A *derivative* is a security with a payoff that depends on the price of another security:
  - The other security is called the *underlying security*
  - Some examples are options, futures, and swaps

- Derivatives are used for different purposes, such as:
  - Hedging and risk management
  - Portfolio insurance
  - Fine tuning a portfolio
  - Speculation

Options

- There are mainly 2 kinds of options:
  - *Call Options*, which allow the holder to buy an underlying asset at a predefined strike price
    ➔ Protects the holder
  - *Put Options*, which allow the holder to sell an underlying asset at a predefined strike price

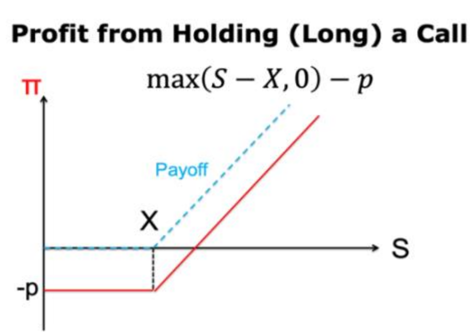
- Furthermore, Options can be classified as *European* or *American Options*:
  - *European Options* allow the holder to exercise the option only at expiration date
  - *American Options* allow the holder to exercise the option anytime until expiration

- For a **call option** with strike price X and stock price S_T, its value will be:

$$C_T = \begin{cases} S_T - X & \text{if } S_T > X \\ 0 & \text{if } S_T \leq X \end{cases}$$

- If the stock price is higher than the strike price, the holder will be able to buy the asset for the lower strike price, and resell it on the market for the higher price
  ➔ In this case, the option is **in the money**

- When the strike price is higher than the stock price, the holder will not exercise it because it would make no sense buying an asset for a higher price than the market price
  ➔ In this case, the option is **out of the money**

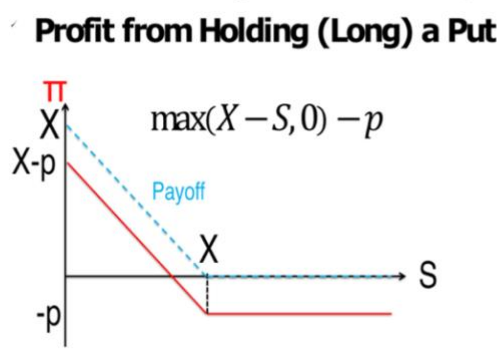


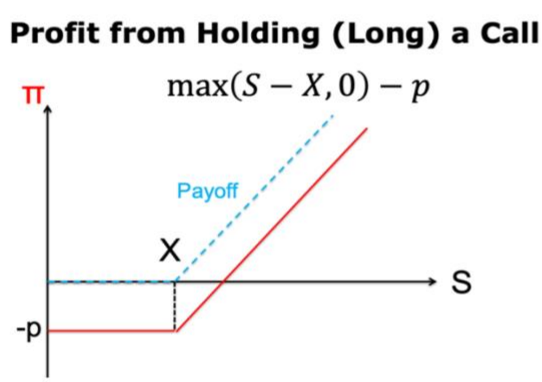
➢ For a **put option** with strike price X and stock price S_T, its value will be:

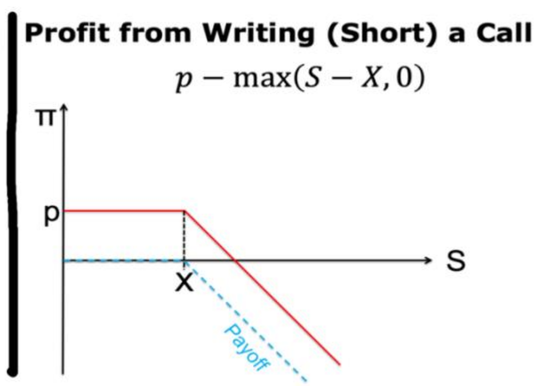
$$P_T = \begin{cases} 0 & \text{if } S_T \geq X \\ X - S_T & \text{if } S_T < X \end{cases}$$

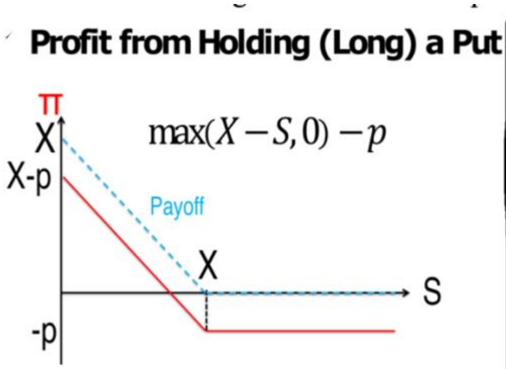
- If the strike price is higher than the stock price, the holder will exercise the option because it is able to sell the underlying asset for a price higher than the market price
  - ➔ The option is **in the money**

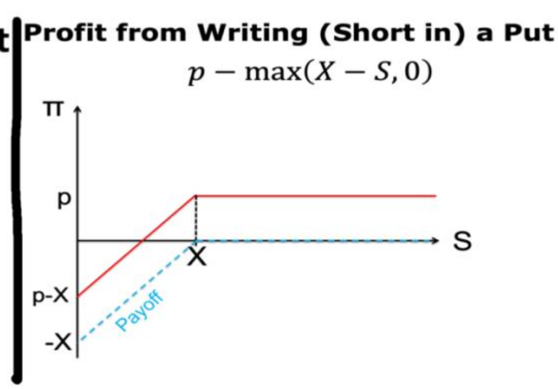
- If the strike price is lower than the strike price, the holder will not exercise the option because, if he did, he would sell the underlying asset for a price lower than the market price
  - ➔ The option is **out of the money**











Options Strategies

1) Using calls for leverage

Example:

- Microsoft share price is S_0=\$80
- A call option with X=\$80 and 6-month maturity costs C_0=\$10
- Risk free rate is 0%

You have \$8,000 and consider 4 strategies

A. Buy 100 shares of Microsoft
B. Buy 800 shares, financed by borrowing \$56,000
C. Buy 800 call options
D. Buy 100 call options and invest \$7000 at the risk-free rate

Strategy	**Stock price at maturity**						
	60	**70**	**80**	**90**	**100**	**110**	**120**
A	-2000	-1000	0	1000	2000	3000	4000
B	-16000	-8000	0	8000	16000	24000	32000
C	-8000	-8000	-8000	0	8000	16000	24000
D	-1000	-1000	-1000	0	1000	2000	3000

2) Protective Put

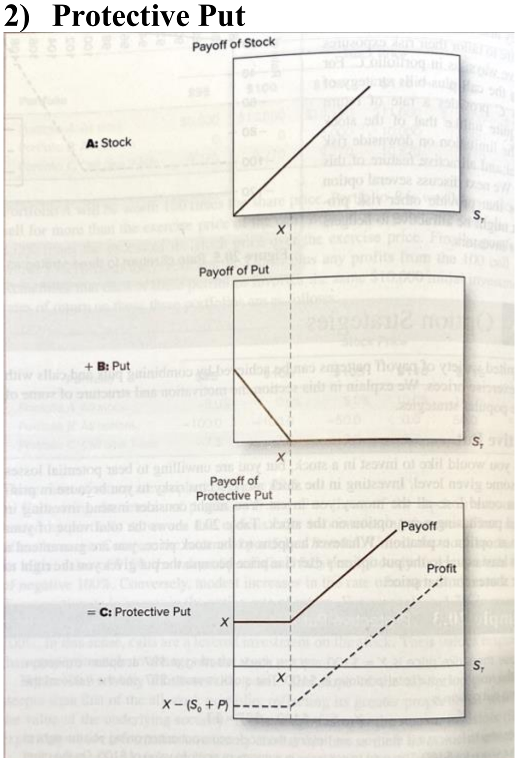


Figure 20.6 Value of a protective put position at option expiration

Assume that an investor is long on a stock, and wants to cover the risk that the stock price will drop:

- In order to do so, he can buy a put option, that will allow him to sell the stock at a strike price
- If he does it, the total profit of the portfolio will be constant until the stock price is greater than the strike price plus the premium paid for the option
- The maximum loss that the investor can bear is equal to the premium paid for the put options

3) Covered Call

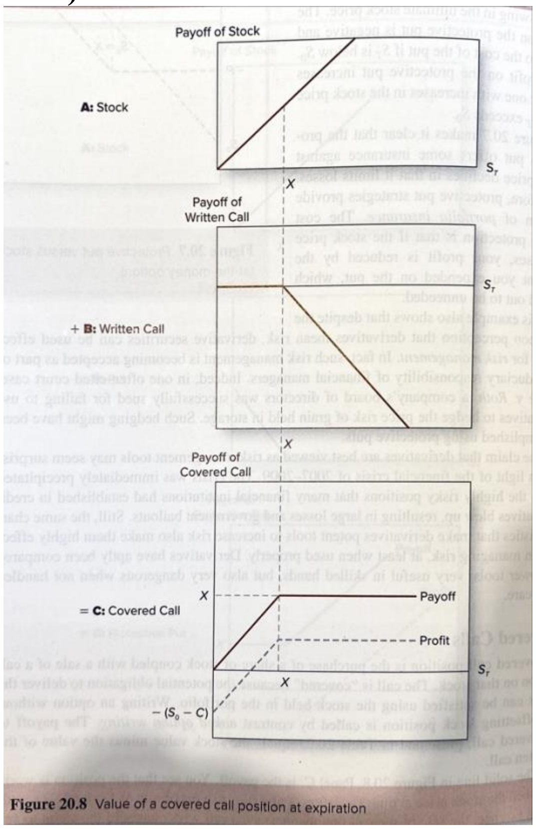


Figure 20.8 Value of a covered call position at expiration

Suppose that an investor sells a call option, because he thinks that out of the money calls are too expensive:

- At the same time, in order to reduce illimited downside risk that shorting a call bears, the investor decides to buy the underlying asset

4) Straddle

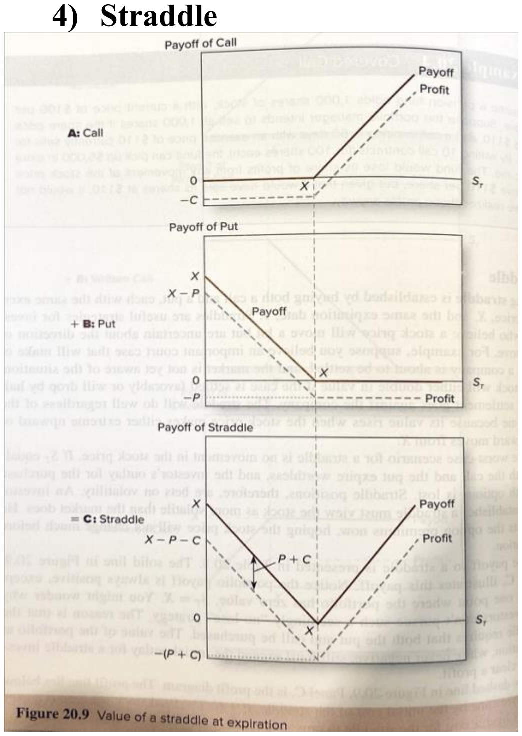


Figure 20.9 Value of a straddle at expiration

Assume that an investor has private information that a particular stock's price will change dramatically soon, but he does not know whether it will increase or fall:

- Assuming that the stock would increase in value, investors could either buy the stock or a call

- Assuming that the stock would decrease in value, investors could either sell the stock, or buy a put

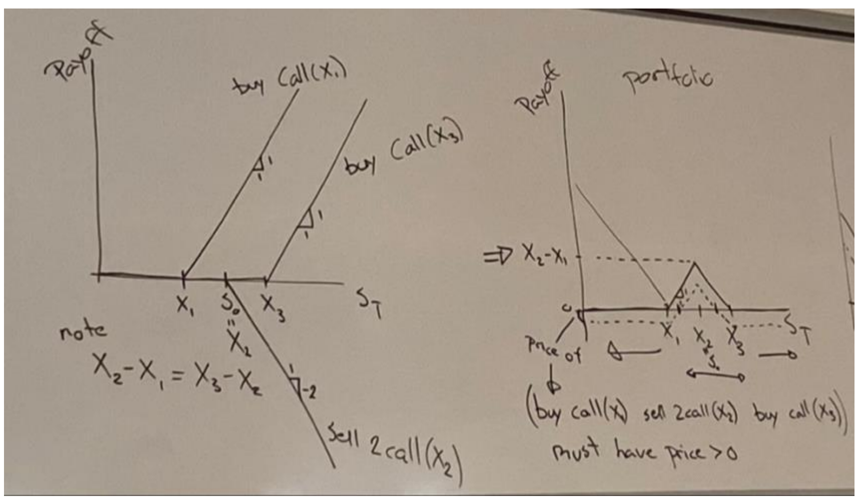
➔ However, since investors do not know the direction of the price change, they will adopt the Straddle Strategy:

- This strategy consists in both buying a put and a call
- If the stock price fall, the put will allow the investor to make a large payoff
- On the other hand, if the stock price rises, the call allows the investor to have a large payoff
- The only way the profit can be negative is if the price does not change dramatically, since the investor will lose the premiums paid to enter the contract of options

5) Butterfly Spread

➢ Assume that you have private information that the volatility of the underlying asset is lower than what the rest of the market believes:

- Therefore, we want to use options to create an asset that profits when the value of the underlying asset moves little
- Buy a call with a strike price of x_1, sell two calls with a strike price of x_2, and buy another call with a strike price of x_3, such that $x_1 < x_2 < x_3$



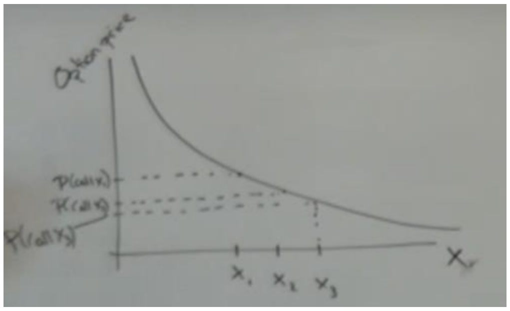
➢ The butterfly strategy allows investors to have positive payoffs if the price of the underlying asset is between x_1 and x_3

- If the stock price drops, or surges, then the investor is going to have a zero payoff
- The maximum payoff possible is the difference between x_1 and x_2

➔ Since the payoff of this strategy is **non-negative**, the price of this strategy must be greater or equal to zero:

$$buy\ call\ (x_1) + sell\ 2\ call\ (x_2) + buy\ call\ (x_3)\ must\ have\ p > 0$$

- From this strategy, we can notice a feature about the curve of call options:
  - As the strike price increases, the option price decreases
  - This is because the higher strike price allows the holder of the option to exercise it for a higher price, and therefore the lower value of the call compensates for it

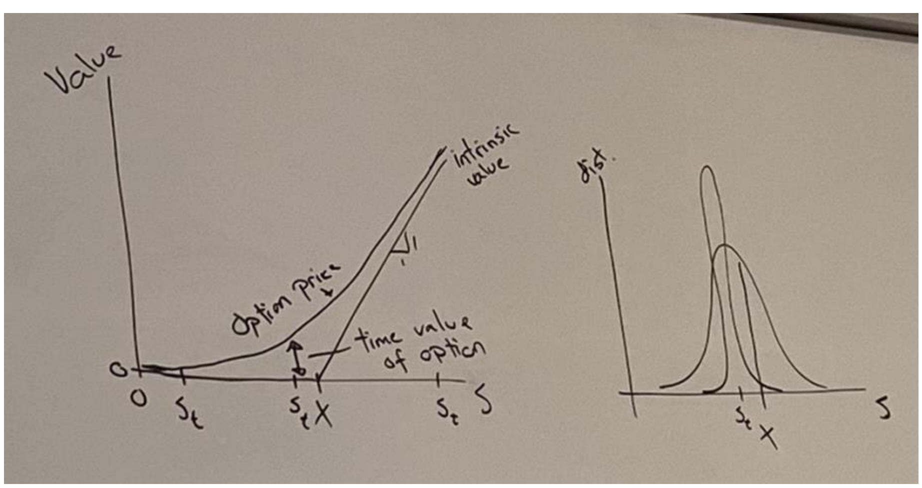


Since the price of a call is a convex function, we can see that the price difference between x_1 and x_2 is larger than the difference between x_2 and x_3

Intrinsic Value of a Call Option

- The intrinsic value of an option is the value of the right to exercise now the option:
  - For an **out-of-the-money option**, the intrinsic value is zero
  - For an **in-the-money option**, the intrinsic value is positive

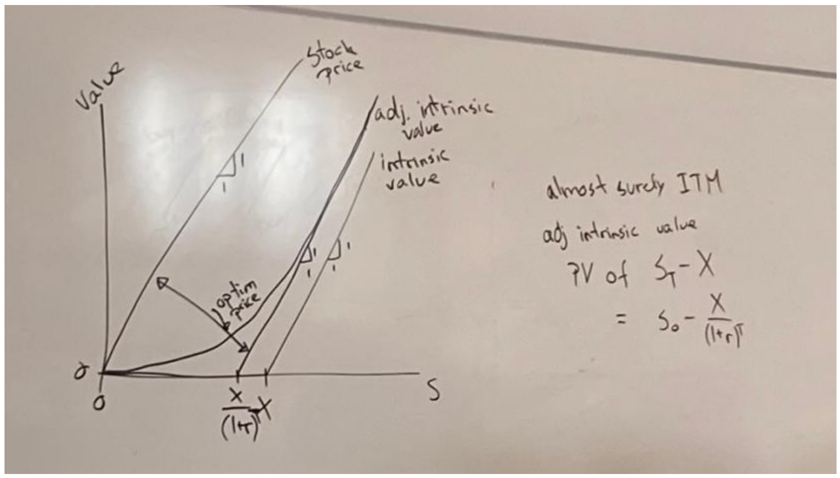
- The difference between the option price and the intrinsic value is called the **time value** of the option
  - An option that is out-of-the-money, and hence has an intrinsic value of zero, may have positive price because of the probability it will be in-the-money in the future
  - For a call option, as the market price increases, the slope of the price will be the same as the slope of the call
  - As the option approaches maturity, the price of the option converges more and more to its intrinsic value
  - The longer the time horizon, the higher the volatility, the higher the probability the option will end up being in-the-money



➢ The **Adjusted Intrinsic Value** is the present value of the strategy to exercise at expiration for sure in the future:

$$\max\left(0, S_0 - \frac{X}{\left(1+R_f\right)^T}\right)$$

➔ Assuming that the stock does not pay dividends before the expiration of the option



- Firstly, we can see that for any value of X, the Adjusted Intrinsic Value will be greater than the intrinsic value of the option
  ➔ For this reason, it is never convenient to exercise a call option before maturity, always assuming that the stock does not pay any dividend

- Secondly, we can see that the Adjusted Intrinsic Value is the lower bound for the call option value:

$$C_0 \geq \max\left(0, S_0 - \frac{X}{\left(1+R_f\right)^T}\right)$$

- Thirdly, the upper value of the option price will be the stock price, because no investor will pay the option more than the underlying asset is actually worth

➔ As the price of the underlying asset increases and the option becomes deeply in-the-money, we can see that the option price almost converges to the Adjusted Intrinsic Value:

- Indeed, when the option is certainly in-the money, the Adjusted Intrinsic Value will be equal to simply the price of the stock today minus the exercise price at expiration, which is equal to the option price

Put-Call Parity

- The *put-call parity* is a fundamental result that states that the difference in prices between a call and a put option is exactly equal to the adjusted intrinsic value of the call:

$$C_0 - P_0 = S_0 - \frac{X}{\left(1 + R_f\right)^T}$$

$$C_0(X) = P_0(X) + S_0 + \frac{X}{\left(1 + r_f\right)^T}$$

  - The left part of the equation is equal to the difference between call and put option prices
  - The right side of the equation is equal to the adjusted intrinsic value of the call option

➔ By the no-arbitrage condition, if we know any three of the four components of the above equation, we can derive the value of the fourth by the no-arbitrage condition

- From the above equation, we can see that the price of the call must be greater than the adjusted intrinsic value of the option:

$$C_0 \geq S_0 - \frac{X}{\left(1 + r_f\right)^T}$$

Example

Suppose a one month call on firm A with strike 95$ is trading at 10$, the similar put with strike 95$ is trading at 7$, the risk-free monthly horizon interest rate is 3%, and the stock is trading at 95$

How can we create an arbitrage strategy?

- Use put-call parity

  C – P = S – X/(1+rf)

  10 – 7 = 95 – 95/1.03

  3 > 2.77

  - In order to exploit the arbitrage opportunity, you can buy the put and sell the call in order to exploit the arbitrage opportunity (sell the expensive one and buy the cheap one)

Put-Call Parity for Stocks with Dividends

- The general *put-call parity* for European options is:

$$C_0 - P_0 = S_0 - \frac{X}{\left(1 + R_f\right)^T} - PV_0(dividends\ before\ time\ T)$$

  - This equation gives a relation between put and call prices, which must hold at all times
  - Hence, if we can price a call, we can automatically know the price of a put

- When companies pay dividends, the stock price falls because one less dividend is discounted in the valuation of the stock:
  - Hence, this is at the same time bad for the call, but good for the put
  - Typically, European options have a short maturity, which make dividends almost certain

Binomial Option Pricing

Two-State Option Pricing

- Assume that the stock price at expiration can only have 2 possible values:
  - Then, compute the option payoff in each scenario and replicate the option payoff with a portfolio of stocks and risk-free securities
  - Compute the price of the replicating portfolio
  - Since we assume no arbitrage, this will be the option price

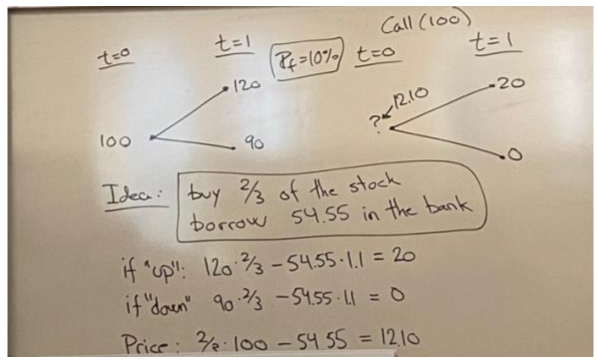
Example

Suppose the stock is currently trading at 100

In one year the stock can trade at either 120 or 90 (and pays no dividends)

The interest rate is 10%

What is the current value of a one year European option with a strike of 100?



- The number of stocks in the replicating portfolio (hedge portfolio) is called the *hedge ratio*, or *Delta*:
  - The hedge ratio tells us how much the option price changes per unit of change in the stock price

$$\Delta = \frac{C^{+} - C^{-}}{S^{+} - S^{-}} = \frac{Call\ in\ the\ money - Call\ out\ of\ the\ money}{Stock\ positive\ case - Stock\ negative\ case}$$

Option Pricing in a Tree

- Assume that the stock price evolves over time in a *tree*, where in each sub-period the stock price can either go up or down
  - In order to deal with this multiple scenario situation, we firstly need to compute the option payoff at expiration in each scenario
  - Then, we should replicate the payoff with a *dynamic hedging strategy* using stocks and risk-free securities
  - In the end, compute the initial price of the replicating strategy, which will be the option price because of the non-arbitrage condition

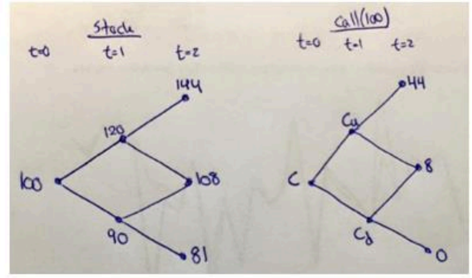
Example

Suppose the stock is currently trading at 100

Every year the stock can go up by a scalar of 1.2 or down by a scalar of 0.9

The interest rate is 10%

What is the current value of a two year European option with a strike of 100?



Cu:
- Delta = (44-8)/(144-108) = 1
- Borrow: 100/1.1 = 90.91
- Price of Cu: 120 – 90.91 = **29.09**

Cd:
- Delta = (8-0)/(108-81) = 0.296
- Borrow: 81*0.296/1.1 = 21.82
- Price of Cd: 90*0.296 – 21.82 = **4.85**

C:
- Delta = (29.09-4.85)/(120-90) = 0.808
- Borrow: (90*0.808-4.85)/1.1 = 61.71
- Price of C: 100*0.808 – 61.71 = **19.10** **Can we arrive at this price in a different way?**

➢ In order to price any derivative security, assume always that the underlying asset evolves over time in a *tree*:
- Then, compute the derivative payoff at expiration in each scenario
- Replicate the derivative payoff with a *dynamic hedging strategy* using the underlying and risk-free securities

➔ In conclusion, compute the initial price of the replicating strategy, which is the derivative's price because of the no arbitrage condition

Risk-Neutral Probabilities in the Binomial Model

➢ We can price derivatives using *price probabilities*:
- In a binomial tree model, the risk-neutral probability of an *up move* is:

$$p = \frac{1 + r_f - down}{up - down}$$

- In this case, *up* and *down* are the states of the world in which the stock market moves either up or down
- This, however, is a risk adjusted probability and does not reflect the true probabilities
- However, we can use this risk-adjusted probabilities to price options

Example

Suppose the stock is currently trading at 100

Every year the stock can go up by a scalar of 1.2 or down by a scalar of 0.9

The interest rate is 10%

Use the risk-neutral probabilities to estimate the current value of a two year European option with a strike of 100?

Black-Scholes-Merton Formula

- *Assumptions*:
  1) The risk-free interest rate is constant and continuously compounded ($\boldsymbol{e^{-\delta t}}$)
  2) The stock price has a *log-normal distribution* to assure that prices are positive, has no jumps, and presents a constant volatility
  3) The stock pays a constant dividend yield
  4) The stock and the risk-free security can be traded at all times with no transaction costs

- If we use the binomial model for European call options, and add more and more scenarios, the option price becomes increasingly more precise:
  - In the limit, we get the **Black-Scholes-Merton Formula** for the price of a European call option:

$$\boldsymbol{C_0 = S_0 e^{-\delta t} N(d_1) - X e^{-rt} N(d_2)}$$

  - The first term of the formula is a function of the price of the underlying asset at time zero multiplied by the factor $\boldsymbol{e^{-\delta t}}$, which takes into account the dividends effect on the price, and $\boldsymbol{N(d_1)}$, which is the risk neutral probability of the stock will end up in the money

  - The second term of the formula is the present value of the strike price multiplied by $\boldsymbol{N(d_2)}$

$d_1 = \dfrac{\ln\left(\frac{S_0}{X}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$	$\boldsymbol{N(d_1)}$ is the risk neutral probability that the option will end up in the money $\boldsymbol{ln\left(\frac{S_0}{X}\right)}$ will be positive when the asset is in-the-money. Therefore, the greater the stock price, the greater the option value since the option will be more surely in the money at maturity $\boldsymbol{\left(r - \delta + \frac{\sigma^2}{2}\right)T}$ is called the *drift* of the price of the asset, which is its expected rate of growth of the price $\boldsymbol{\sigma\sqrt{T}}$ and $\boldsymbol{\frac{\sigma^2}{2}T}$ are adjustments made in order to adjust the expected value of the asset for volatility and time
$d_2 = d_1 - \sigma\sqrt{T}$	$\boldsymbol{N(d_2)}$ is the risk adjusted probability that the strike price will be paid

1) If the option is almost certain to be in-the-money at maturity, then:
   - $N(d_1) \cong N(d_2) \cong 1$
   - The option adjusted intrinsic value will be simply the present value of the stock price minus the present value of the strike price:
$$C_0 = S_0 e^{-\delta T} - X e^{-rT}$$

2) If the option is almost certain to be out-of-the-money at maturity, then:
   - $N(d_1) \cong N(d_2) \cong 0$
   - The option price will be close to zero

➔ In general, according to the model, the price is the risk-adjusted expected payment at maturity

➢ The risk-free rate determines the expected future value of a stock:
   - Hence, after an increase in the risk-free rate, keeping everything else constant, the expected future value of the stock will be higher
   - This is equivalent as saying that the probability distribution of the stock prices shifts to the right, and will be centered at a higher expected future asset value
   - Therefore, after an increase in the risk-free rate, the option price increases

➢ Assuming that dividends and the risk-free rate are both zero, the expected value of the stock price will be equal to the current stock price:
   - If dividends increase, the future stock price will be expected to be lower
   - This is equivalent as saying that the distribution of the stock price will be centered at a lower level of expected stock price
   - Therefore, if a corporation announces dividends, this is generally bad news for call options, as dividends reduce the stock price

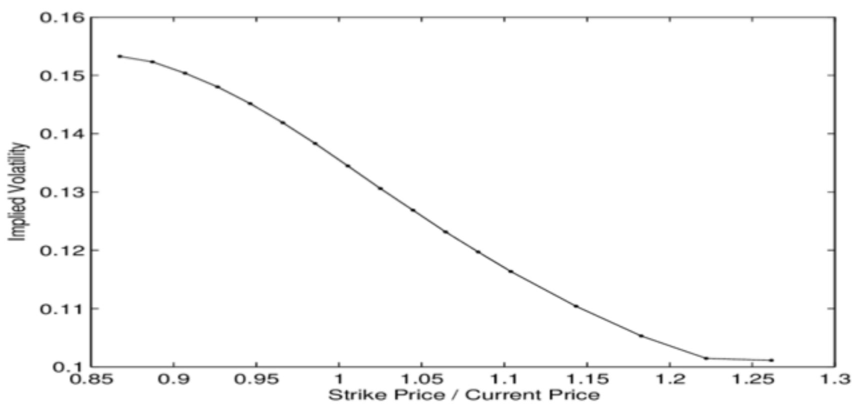
➢ Time has an effect on both volatility and the drift of the distribution of call option:
   - If time increases, the volatility of the distribution will increase, thus increasing the call price
   - Furthermore, when time increases, it also has a positive effect on the drift of the distribution, which will shift to the right
   - However, if we are in a world in which the dividend yield is greater than the risk-free rate, it is not obvious that maturity increases the option value

➢ Put options are:
   - Increasing in volatility
   - Decreasing in risk-free rates
   - Increasing in dividends
   - Decreasing in time (assuming that the risk-free rate is greater than the dividend yield)

Implied volatility

- The *implied volatility relationship* states that:
  - For every level of volatility, σ, there is a corresponding option price, C_0
  - For every option price, C_0, there is a corresponding level of volatility, σ

- According to the Black-Scholes-Merton formula, all options on the same stock, but with different strikes, should be priced with the same implied volatility:
  - Hence, the implied volatility tells that the option price does not depend on the strike price
  - However, empirical evidence on the S&P 500 shows that the implied volatilities in the real world are not constant across strike prices (they are not on a straight line in the graph below)
  - The *volatility smile* means that the distribution of stock returns is more or less skewed than what the Black-Scholes-Merton model suggests



- This empirical evidence shows us that the model does not hold and what are the assumptions that make it not valid:
  - However, it provides a great intuition behind option pricing models

Hedge Ratio or Delta

- The Delta is the number of stocks in a replicating portfolio, and is the change in the value of the call after a change in value of the underlying asset:

$$\Delta = \frac{\partial C_0}{\partial S_0} = e^{-\delta t} N(d_1)$$

  - If an investment bank writes and option to a client, it will hedge the position by buying Δ shares
  - Since Δ is changing over time, the bank must keep adjusting the number of shares held using a *dynamic hedging strategy*

Black-Scholes-Merton Pricing of a Put Option

- Using the put-call parity, we can use the model to price a European put option:
  - *Put-Call Parity*

$$P_0 = C_0 - S_0 + PV_0(X) + PV_0(dividends\ before\ T) =$$

$$= C_0 - S_0 + Xe^{-rt} + S_0\left(1 - e^{-\delta t}\right) =$$

$$= \boldsymbol{Xe^{-rt}\left(1 - N(d_2)\right) - S_0 e^{-\delta t}\left(1 - N(d_1)\right)}$$

- $\boldsymbol{\left(1 - N(d_2)\right)}$ and $\boldsymbol{\left(1 - N(d_1)\right)}$ are the *risk-neutral probabilities* that the put option will end up in the money:
  - Recall that $\boldsymbol{N(d_1)}$ and $\boldsymbol{N(d_2)}$ are the risk-neutral probabilities that the call will end up in the money
  - Recall that the put is in the money when the call is not